PROPAGATION OF MAGNETOELASTIC STRESS WAVES FROM A CYLINDRICAL CAVITY IN A CONDUCTING MEDIUM

I. T. Selezov

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We study the influence of a constant axial magnetic field on the propagation of magnetoelastic compression waves from a cavity containing a magnetoacoustic medium with a jump of the surface force given at the wall. The problem is examined in [1] in the case in which there is a vacuum in the cavity and an ideal conductor outside, without any study of the effect of a magnetic field.

Here we examine the problem for both weak and ideal conductivities. The equations are linearized and Laplace transformed. Approximate asymptotic solutions are constructed which are valid in the vicinity of the wave fronts. The solutions are studied analytically and numerically.

1. If we assume that the conditions of elastic isotropicity, geometric and elastic linearity, and isotropicity of the dielectric and magnetic permeabilities are satisfied for a magnetoelastic medium, then the system of magnetoelasticity equations is

$$G\nabla^2\mathbf{u} + (\lambda + G)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \rho_c \frac{\partial^2\mathbf{u}}{\partial t^2} - \mathbf{j}\times\mathbf{B} - \rho_e\mathbf{E} - \mathbf{F}, \tag{1.1}$$

$$\mathrm{rot}\,\mathbf{H} = \mathbf{j} + \frac{\partial\mathbf{D}}{\partial t}, \quad \mathrm{rot}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \mathrm{div}\,\mathbf{B} = 0, \quad \mathrm{div}\,\mathbf{D} = \rho_e, \tag{1.2}$$

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\partial\mathbf{u}}{\partial t}\times\mathbf{B}\right), \qquad \mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{D} = \varepsilon\mathbf{E}. \tag{1.3}$$

Under analogous assumptions the equations of a magnetoacoustic medium have the form

$$-\mathrm{grad}\,p = \rho_0\frac{\partial\mathbf{v}}{\partial t} - \mathbf{j}^a\times\mathbf{B}^a - \rho_e{}^a\,\mathbf{E}^a - \mathbf{F}^a, \tag{1.4}$$

$$\frac{\partial\rho}{\partial t} + \rho_0\,\mathrm{div}\,\mathbf{v} = 0, \qquad p = p(\rho). \tag{1.5}$$

The system (1.4), (1.5) is also supplemented by equations analogous to (1.2), (1.3).

The conservation laws lead to the following boundary conditions at the interface of the two media:

$$\frac{\partial u_n}{\partial t} = v_n, \quad (\sigma_{ik} + T_{ik})\,n^i = (\sigma_{ik}{}^a + T_{ik}{}^a)\,n^i + Q_k \quad (i,\ k = 1,\ 2,\ 3), \tag{1.6}$$

$$\mathbf{n}\cdot(\mathbf{B}-\mathbf{B}^a) = 0, \quad \mathbf{n}\times(\mathbf{H}-\mathbf{H}^a) = 0, \quad \mathbf{n}\cdot(\mathbf{D}-\mathbf{D}^a) = 0, \quad \mathbf{n}\times(\mathbf{E}-\mathbf{E}^a) = 0. \tag{1.7}$$

Here

$$\sigma_{ik} = c_{ikjl}\frac{1}{2}\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right), \quad T_{ik} = \varepsilon E_i E_k + \mu H_i H_k - \frac{1}{2}\delta_{ik}(\varepsilon E^2 + \mu H^2).$$

In the following we assume that there are no mass forces ($\mathbf{F} = \mathbf{F}^a = 0$) or free electrical charges ($\rho_e = \rho_e^a = 0$) in either of the media. In the magnetoelastic medium we also neglect the displacement currents ($\partial D/\partial t = 0$), and we consider the magnetoacoustic medium to be nonconducting ($\sigma^a = 0$) and isentropic. We represent the magnetic field as the sum of the unperturbed and perturbed components $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, where $|\mathbf{h}| \ll |\mathbf{H}_0|$, which permits linearization of all the equations.

With account for these assumptions, after some transformations we obtain from (1.1)–(1.3) for the magnetoelastic medium

$$G\nabla^2\mathbf{u} + (\lambda + G)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \rho_c\frac{\partial^2\mathbf{u}}{\partial t^2} - \mu(\mathrm{rot}\,\mathbf{h})\times\mathbf{H}_0, \tag{1.8}$$

$$\frac{\partial\mathbf{h}}{\partial t} = \frac{1}{\mu\sigma}\nabla^2\mathbf{h} + \mathrm{rot}\left(\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}_0\right), \qquad \mathrm{div}\,\mathbf{h} = 0, \tag{1.9}$$

and from (1.4) and (1.5) for the magnetoacoustic medium

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0, \qquad \left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\mathbf{h}^a\\ \mathbf{e}^a\end{Bmatrix} = 0, \tag{1.10}$$

$$\operatorname{rot}\mathbf{h}^a = \varepsilon\frac{\partial \mathbf{e}^a}{\partial t}, \quad \operatorname{rot}\mathbf{e}^a = -\mu\frac{\partial \mathbf{h}^a}{\partial t}, \quad \mathbf{v} = -\operatorname{grad}\varphi, \quad p = \rho_0\frac{\partial\varphi}{\partial t}. \tag{1.11}$$

For weak conductivity all the quantities can be expanded in terms of the small magnetic Reynolds number R_m and we can retain in the equations terms of first order of smallness [2]. In this approximation the induced magnetic fields are quantities of higher order of smallness [3]. In this case the last term in (1.8) and the first equation of (1.9) take the form

$$-\mu\sigma\left(\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}_0\right)\times\mathbf{H}_0, \qquad \operatorname{rot}\mathbf{h} = \frac{\mu\sigma}{R_m}\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}_0. \tag{1.12}$$

For ideal conductivity the last term in (1.8), Eq. (1.9), and the first equation of (1.3) reduce to the form

$$-\mu\,[\operatorname{rot}\operatorname{rot}(\mathbf{u}\times\mathbf{H}_0)]\times\mathbf{H}_0, \quad \mathbf{h} = \operatorname{rot}(\mathbf{u}\times\mathbf{H}_0), \quad \mathbf{e} = -\mu\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}_0. \tag{1.13}$$

2. Let us examine in the r, θ, z cylindrical coordinate system a magnetoelastic medium with a cylindrical cavity of radius a, filled with a magnetoacoustic medium, subjected to the undisturbed magnetic field $(0, 0, B_{0z})$ and the action on the cavity wall of the load $Q_r = -q_0 s(t)$, where s(t) is the Heaviside function.

Introducing dimensionless parameters by the formulas

$$(r^*, u_r^*) = \frac{1}{a}(r, u_r), \quad t^* = \frac{c_e}{a}t, \quad (v^*, c^*) = \frac{1}{c_e}(v, c),$$

$$(\sigma_{ik}^*, q_0^*) = \frac{1}{\lambda+2G}(\sigma_{ik}, q_0), \quad \rho^* = \frac{\rho}{\rho_c}, \quad \varphi^* = \frac{\varphi}{c_e a}, \quad h_i^* = \frac{h_i}{H_{0z}},$$

$$e_i^* = \frac{e_i}{\mu H_{0z} c_e}, \quad R_m = \mu\sigma c_e a, \quad P_H = \frac{\mu H_{0z}^2}{\lambda+2G}$$

and dropping the asterisks, we obtain from (1.10) and (1.11) in the region $0 < r < a$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0, \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)h_z^a = 0, \tag{2.1}$$

$$-\frac{\partial h_z^a}{\partial r} = \frac{1}{c_1^2}\frac{\partial e_\theta^a}{\partial t}, \quad \frac{1}{r}\frac{\partial}{\partial r}re_\theta^a = -\frac{\partial h_z^a}{\partial t}. \tag{2.2}$$

From (1.8), (1.9), (1.12), (1.13) in the region $a \le r \le \infty$ we obtain, for weak conductivity, that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial r} - \frac{1}{r}\right)u_r = \left(\frac{\partial^2}{\partial t^2} + R_m P_H\frac{\partial}{\partial t}\right)u_r, \quad \frac{\partial h_z}{\partial r} = \frac{\partial u_r}{\partial t} \tag{2.3}$$

and for ideal conductivity that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u_r = \frac{1}{1+P_H}\frac{\partial^2 u_r}{\partial t^2}, \quad h_z = -\frac{1}{r}\frac{\partial}{\partial r}ru_r, \quad e_\theta = \frac{\partial u_r}{\partial t}. \tag{2.4}$$

From (1.6) and (1.7) we find the linearized boundary conditions at the surface r = 1 (all the conditions other than those which are satisfied identically are presented):

$$\frac{\partial u_r}{\partial r} + \frac{\lambda}{\lambda+2G}u_r =$$

$$= \rho_0\frac{\partial\varphi}{\partial t} = q_0 s(t) + \frac{1}{2}P_H\left(1 - \frac{\mu^a}{\mu}\right) + P_H\left(R_m h_z - \frac{\mu^a}{\mu}h_z^a\right), \tag{2.5}$$

$$-\frac{\partial\varphi}{\partial r} = \frac{\partial u_r}{\partial t}, \quad R_m h_z = h_z^a. \tag{2.6}$$

In the case of ideal conductivity the last term in (2.5) and the second equation of (2.6) are replaced by

$$P_H\left(h_z - \frac{\mu^a}{\mu} h_z^a\right), \qquad e_\theta = e_\theta^a. \tag{2.7}$$

Moreover, the unknown functions must satisfy the radiation conditions and be bounded as $r \to \infty$.

Zero initial conditions are assumed.

We note that the first equations in (2.3) and (2.4), which contain terms that reflect the influence of the magnetic field, are independent, i. e., the systems of equations in each of the considered approximations broke down partially. But in boundary conditions (2.5)–(2.7) the connection between the elastic and electromagnetic fields remains.

In the following we examine nonmagnetizable media $\mu = \mu^a$; then the boundary conditions simplify. We see from conditions (2.5) and (2.6) that in the case of weak conductivity the connection between the fields disappears in the boundary conditions as well, i. e., we can determine independently the field of the elastic variables and then the electromagnetic field. An analogous simplification also takes place in MHD [2]. We see from (2.7) that for ideal conductivity with $\mu = \mu^a$ the connection between the fields remains in the boundary conditions.

3. Equations (2.1)–(2.7) are Laplace transformed

$$F(r, \varkappa) = \int_0^\infty f(r, t)\exp(-\varkappa t)\,dt.$$

The solutions in transform space, satisfying the boundary and initial conditions, are written in the case of weak conductivity as

$$\Phi = q_0 c_0 \frac{I_0(\varkappa r/c_0)}{\varkappa \Delta I_1(\varkappa/c_0)} \qquad (0 < r < a), \tag{3.1}$$

$$U_r = -q_0 \frac{K_1(\Omega r)}{\varkappa \Delta K_1(\Omega)}, \quad \Sigma_{rr} = -\frac{q_0}{\varkappa \Delta K_1(\Omega)}\left[\Omega K_1'(\Omega r) + \frac{\lambda}{\lambda+2G}\frac{1}{r}K_1(\Omega r)\right] \quad (a \leqslant r \leqslant \infty). \tag{3.2}$$

Here $I_0(z)$ is the modified Bessel function and $K_1(z)$ is the Macdonald function,

$$\Omega = \sqrt{\varkappa^2 + R_m P_H \varkappa}, \quad \Delta = \frac{\lambda}{\lambda+2G} + \Omega\frac{K_1'(\Omega)}{K_1(\Omega)} - \rho_0 c_0 \varkappa \frac{I_0(\varkappa/c_0)}{I_1(\varkappa/c_0)}. \tag{3.3}$$

In the case of ideal conductivity

$$(0 < r < a) \qquad (a \leqslant r \leqslant \infty)$$

$$\Phi = -q_0 c_0 \frac{I_0(\varkappa r/c_0)}{\varkappa \Delta_1 I_1(\varkappa/c_0)}, \quad U_r = q_0 \frac{K_1(\varkappa_1)}{\varkappa \Delta_1} K_1(\varkappa_1 r),$$

$$H_z^a = -\frac{q_0}{c_1}\frac{I_0(\varkappa r/c_1)}{\Delta_1 I_1(\varkappa/c_1)}, \quad H_z = \frac{q_0}{\sqrt{1+P_H}}\frac{K_1(\varkappa_1)}{\Delta_1} K_0(\varkappa_1 r), \tag{3.4}$$

$$E_\theta^a = q_0 \frac{I_1(\varkappa r/c_1)}{\Delta_1 I_1(\varkappa/c_1)}, \quad E_\theta = q_0 \frac{K_1(\varkappa_1)}{\Delta_1} K_1(\varkappa_1 r),$$

$$\Sigma_{rr} = -q_0 \frac{K_1(\varkappa_1)}{\varkappa \Delta_1}\left[\varkappa_1 K_0(\varkappa_1 r) + \frac{2G}{\lambda+2G}\frac{1}{r}K_1(\varkappa_1 r)\right], \tag{3.5}$$

where

$$\varkappa_1 = \frac{\varkappa}{\sqrt{1+P_H}}, \quad \Delta_1 = \frac{2G}{\lambda+2G} + \varkappa\sqrt{1+P_H}\frac{K_0(\varkappa_1)}{K_1(\varkappa_1)} + \rho_0 c_0 \varkappa \frac{I_0(\varkappa/c_0)}{I_1(\varkappa/c_0)} + P_H \frac{\varkappa}{c_1}\frac{I_0(\varkappa/c_1)}{I_1(\varkappa/c_1)}.$$

We note that in the expressions for Δ and Δ_1, appearing in (3.1)–(3.5), the first and second terms characterize the magnetoelasticity, the third term characterizes the acoustics, and the fourth term characterizes the electromagnetic field in the acoustic medium.

It is very difficult to construct the exact transformation of the resulting solutions with the aid of the Riemann–Mellin integral. Therefore, in the following we construct approximate asymptotic solutions which are valid near the front of the magnetoelastic wave.

The cylindrical functions appearing in (3.1)–(3.5) are replaced by their asymptotic representations for large $\varkappa$, in which two terms are retained. Retaining in (3.1)–(3.5) all terms up to and including order $1/\varkappa$ and making estimates of the dropped terms, we obtain the approximate values of the transforms, from which we then find the originals. We present the final expressions for the radial stresses σ_{rr}.

For the weakly conducting medium

$$\frac{\sigma_{rr}}{q_0} \approx -\frac{\exp\left[-{}^1\!/_2 R_m P_H (r-1)\right]}{(1+\rho_0 c_0)\sqrt{r}} \{s[t-(r-1)] + (A_0 + A_1 r^{-1})[t-(r-1)]\}; \text{ for } t > (r-1) \geqslant 0$$

$$\sigma_{rr}/q_0 = 0 \text{ for } t < (r-1). \tag{3.6}$$

For the ideally conducting medium

$$\frac{\sigma_{rr}}{q_0} \approx -\frac{1}{A\sqrt{r}}\left\{\frac{-\lambda+14G}{\lambda+2G}\frac{1}{r} + \right.$$

$$\left. + \left(\frac{A}{B\sqrt{1+P_H}} - \frac{-\lambda+14G}{\lambda+2G}\right)\exp\left[-\frac{A}{B}\left(t - \frac{r-1}{\sqrt{1+P_H}}\right)\right]\right\} \text{ for } t > \frac{r-1}{\sqrt{1+P_H}} \geqslant 0;\ \sigma_{rr}/q_0 = 0 \text{ for } t < \frac{r-1}{\sqrt{1+P_H}}. \tag{3.7}$$

Here

$$A_0 = \frac{{}^3\!/_8 c_0 - {}^1\!/_2 \rho_0 c_0^2 - {}^7\!/_8 + \lambda/(\lambda+2G) + {}^1\!/_2 R_m R_H}{1+\rho_0 c_0} - \frac{3}{8}c_0, \quad A_1 = \frac{7}{8} - \frac{\lambda}{\lambda+2G}$$

$$A = -\frac{1}{8} + \frac{2G}{\lambda+2G} + \frac{1}{2}\rho_0 c_0^2\left(1 + \frac{3}{4}\frac{\sqrt{1+P_H}}{c_0}\right) + \frac{3}{8}P_H\left(1 + \frac{\sqrt{1+P_H}}{c_1}\right)$$

$$B = \sqrt{1+P_H} + \rho_0 c_0 + \frac{P_H}{c_1}. \tag{3.8}$$

Solution (3.6) was obtained with the asymptotic error $[t-(r-1)] \leqslant {}^3\!/_8\,\varepsilon$, where ε is the specified computational accuracy. Solution (3.7) is valid in a very small vicinity of the wave front. As $[t-(r-1)] \to 0$ in (3.6) and $[t-(r-1)]/\sqrt{1+P_H} \to 0$ in (3.7), these solutions become exact for the wave fronts.

4. Let us investigate the influence of the magnetic field on σ_{rr}. We see from (3.6) in the case of weak conductivity that for $r > 1$ with increase of R_m and P_H the quantity $|\sigma_{rr}|$ decreases exponentially. Behind the wave front there is some increase of the stresses, which follows from (3.8): the quantity A_0 increases with increase of $R_m P_H$.

In the ideal conductivity case we first of all estimate the influence of the electromagnetic field of the magnetoacoustic medium. In expressions (3.8) for A and B there appear the terms $\sqrt{1+P_H}/c_1$ and P_H/c_1, which we write in the form $c_e/c_1\sqrt{1+P_H}$ and $c_e/c_1 P_H$ and compare with unity. Since the strong constant magnetic fields which can be created in laboratory conditions at the present time are of the order of 10 tesla [4, 5], these terms will be small quantities of higher order. This implies that in the case in question in an acoustic medium (or vacuum) we can neglect the displacement currents if we are examining slow motions in the adjoining magnetoelastic medium.

In the case of ideal conductivity, increase of P_H leads to increase of the wave front propagation velocity by a factor of $\sqrt{1+P_H}$ and reduction of the wave front amplitude by a factor of $(B\sqrt{1+P_H})^{-1}$; we see from (3.8) that the quantity B increases.

The presence of the acoustic medium reduces the stresses in both cases, which is a result of its inertial and elastic properties.

5. As examples we examine a weakly conducting medium of the bismuth type with conductivity σ which is 100 times less, so that the condition $R_m < 1$ is satisfied, and an ideally conducting medium of the copper type with conductivity $\sigma = \infty$. In the cavity there is an acoustic medium with the properties of water; the parameters are:

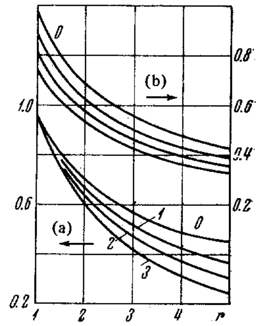
in the first case (a)

$$\rho_0/\rho_c = 0.102, \quad c_0/c_e = 0.663, \quad \nu = 0.33;$$

in the second case (b)

$$\rho_0 / \rho_c = 0.113, \; c_0 / c_e = 0.333, \; c_1 / c_e = 0.686 \cdot 10^5, \; \nu = 0.34 .$$

The figure shows the radial stresses σ_{rr}, normalized in accordance with (3.6) and (3.7), at the wave fronts as a function of the distance r/a for the weakly and ideally conducting media. The numerals 0, 1, 2, 3 denote the values 0, 0.1, 0.2, 0.3 of the quantities $R_m P_H$ or P_H. These values correspond to strong constant magnetic fields which are an order higher than those obtained in laboratories [4, 5].



6. Analysis of solutions (3.6) and (3.7) and the calculations makes it possible to draw the following conclusions:

a) increase of the magnetic field intensity H_{0z} leads to reduction of $|\sigma_{rr}|$ at the wave front in the case of both weak and ideal conductivity;

b) behind the wave front the magnetic field causes a small increase of $|\sigma_{rr}|$ in both cases;

c) the displacement currents in the acoustic medium (or in a vacuum) are small quantities of higher order if $B \leq 10^4$ tesla and in the adjacent magnetoelastic medium we examine processes with velocities which do not exceed the velocity of expansion waves in the elastic medium;

d) the presence of the acoustic medium in the cavity leads to a reduction of $|\sigma_{rr}|$;

e) the influence of the constant magnetic field on the magnetoelastic stresses in nonmagnetizable media can be detected in superstrong magnetic fields $B \geq 10^2$ tesla.

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